MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 15: Generating Functions I: Generalized Binomial Theorem and Fibonacci Sequence

In this lectures we start our journey through the realm of generating functions. Roughly speaking, a generating function is a formal Taylor series centered at 0, that is, a formal Maclaurin series. In general, if a function $f(x)$ is smooth enough at $x = 0$, then its Maclaurin series can be written as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \tag{0.1}$$

where $f^{(n)}(x)$ is the n-th derivative of $f(x)$. We know from Calculus that the Maclaurin series of the function $(1-x)^{-1}$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \tag{0.2}$$

The Maclaurin series of every polynomial function is itself. In particular, the Binomial Theorem gives us an explicit formula for the Maclaurin series/polynomial of any nonnegative integer power of the binomial $1 + x$:

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n.$$

But what if we want to compute the Maclaurin series of $(1+x)^r$ when r is not a nonnegative integer?

Generalized Binomial Theorem. The Generalized Binomial Theorem allows us to express $(1+x)^r$ as a Maclaurin series using a natural generalization of the binomial coefficients. For any $r \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we set

$$\binom{r}{n} := \frac{r(r-1)\cdots r-n+1}{n!}. \tag{0.3}$$

Observe that when $r \in \mathbb{N}_0$, we recover the standard formula for the binomial coefficients. We are now in a position to generalize the Binomial Theorem.

Theorem 1. *For any $r \in \mathbb{R}$,*

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n. \tag{0.4}$$

Proof. Set $f(x) = (1+x)^r$. For each $n \in \mathbb{N}_0$, we see that $f^{(n)}(x) = (r)_n(1+x)^{r-n}$, and so $f^{(n)}(0)/n! = \binom{r}{n}$. Therefore the Maclaurin formula of $f(x)$ is that one in the right-hand side of (0.4). □

As an application of Theorem 1, we can generalize (0.2).

Example 2. Let us find the Maclaurin series of $(1-x)^{-m}$ when $m \in \mathbb{N}$. First, note that for each $n \in \mathbb{N}_0$,

$$\begin{aligned}\binom{-m}{n} &= \frac{1}{n!}\prod_{i=0}^{n-1}(-m-i) = \frac{(-1)^n}{n!}m(m+1)\cdots(m+n-1) \\ &= (-1)^n\frac{(m+n-1)!}{n!(m-1)!} = (-1)^n\binom{m+n-1}{m-1}.\end{aligned}$$

Now in light of Theorem 1,

$$(1+x)^{-m} = \sum_{n=0}^{\infty}\binom{-m}{n}x^n = \sum_{n=0}^{\infty}(-1)^n\binom{m+n-1}{m-1}x^n = \sum_{n=0}^{\infty}\binom{m+n-1}{m-1}(-x)^n.$$

Evaluating the previous identity at $-x$, we obtain that

$$(1-x)^{-m} = \sum_{n=0}^{\infty}\binom{m+n-1}{m-1}x^n.$$

Generating Function of a Sequence. We can associate to any sequence $(a_n)_{n\geq 0}$ of real numbers the formal power series $\sum_{n=0}^{\infty} a_n x^n$. We call this formal power series the (*ordinary*) *generating function* of the sequence $(a_n)_{n\geq 0}$. When $\sum_{n=0}^{\infty} a_n$ converges to a function $F(x)$ in some neighborhood of 0, we also call $F(x)$ the (*ordinary*) *generating function* of $(a_n)_{n\geq 0}$.

Example 3. The generating function of a sequence $(a_n)_{n\geq 0}$ satisfying that $a_n = 0$ for every $n > d$ is the polynomial $\sum_{n=0}^{d} a_n x^n$.

Example 4. It follows from (0.2) that $(1-x)^{-1}$ is the generating function of the constant sequence all whose terms equal 1.

Example 5. For each $m \in \mathbb{N}$, we have seen in Example 2 that the generating function of the sequence $\left(\binom{m+n-1}{m-1}\right)_{n\geq 0}$ is $(1-x)^{-m}$.

We can actually use generating functions to find explicit formulas for linear recurrence relations. The following example illustrates how to do this.

Example 6. Consider the sequence $(a_n)_{n\ge 0}$ recurrently defined as follows: $a_0 = 2$ and $a_{n+1} = 5a_n$ for every $n \in \mathbb{N}_0$. Let us find a closed formula for a_n. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $(a_n)_{n\ge 0}$. Since $\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} 5a_n x^n$, we see that $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = 5x\sum_{n=0}^{\infty} a_n x^n$ and, therefore,

$$F(x) = 2 + \sum_{n=1}^{\infty} a_n x^n = 2 + 5x\sum_{n=0}^{\infty} a_n x^n = 2 + 5xF(x).$$

Hence $F(x) = 2(1-5x)^{-1}$, and so

$$\sum_{n=0}^{\infty} a_n x^n = F(x) = \frac{2}{1-5x} = 2\sum_{n=0}^{\infty}(5x)^n = \sum_{n=0}^{\infty} 2\cdot 5^n x^n,$$

from which we can obtain the desired explicit formula for a_n, namely, $a_n = 2\cdot 5^n$ for every $n \in \mathbb{N}_0$.

Recall that the Fibonacci sequence is defined by the recurrence $F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$ and $F_1 = 1$. Let us conclude this lecture providing an explicit formula for the Fibonacci numbers.

Example 7. Let $F(x)$ be the generating function of the Fibonacci sequence. Then

$$F(x) - x = \sum_{n=1}^{\infty} F_{n+1}x^{n+1} = x\sum_{n=1}^{\infty} F_n x^n + x^2\sum_{n=1}^{\infty} F_{n-1}x^{n-1} = xF(x) + x^2F(x).$$

Solving for $F(x)$, we obtain that

$$F(x) = -\frac{x}{x^2+x-1} = -\Big(\frac{A}{x-\alpha} + \frac{B}{x-\beta}\Big),$$

for some $A, B \in \mathbb{R}$, where α and β are the real roots of $x^2 + x - 1$. From $x = A(x-\beta) + B(x-\alpha)$, we can readily deduce that $A = \frac{\alpha}{\alpha-\beta}$ and $B = \frac{\beta}{\beta-\alpha}$. Thus,

$$\begin{aligned} F(x) &= \frac{A}{\alpha-x} + \frac{B}{\beta-x} = \frac{1}{\alpha-\beta}\Big(1-\frac{x}{\alpha}\Big)^{-1} + \frac{1}{\beta-\alpha}\Big(1-\frac{x}{\beta}\Big)^{-1} \\ &= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\Big(\frac{x}{\alpha}\Big)^n + \frac{1}{\beta-\alpha}\sum_{n=0}^{\infty}\Big(\frac{x}{\beta}\Big)^n = \sum_{n=0}^{\infty}\Big(\frac{\alpha^{-n}}{\alpha-\beta} + \frac{\beta^{-n}}{\beta-\alpha}\Big)x^n. \end{aligned}$$

Taking $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$, we obtain the following explicit formula:

$$F_n = \frac{1}{\sqrt{5}}\Big(\frac{2}{-1+\sqrt{5}}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{2}{-1-\sqrt{5}}\Big)^n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{\sqrt{5}}\Big(\frac{1-\sqrt{5}}{2}\Big)^n.$$

Practice Exercises

Exercise 1. *Consider the sequence* $(a_n)_{n \ge 0}$ *satisfying that* $a_0 = 3$ *and* $a_{n+1} = 5a_n + 7^n$ *for every* $n \in \mathbb{N}_0$. *Find an explicit formula for* a_n.

Exercise 2. *Find a closed form for the generating function of the sequence* $(n^2)_{n \ge 0}$.

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